Comparing the Horvitz-Thompson estimator and Hajek estimator

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Consider a finite population with labels $U = \{1, 2, \dots, N\}$. Suppose $y_k, k \in U$ are values of the variable of interest in the population. We wish to estimate the total $\sum_{k=1}^{N} y_k$ using a sample s selected from the population U. Assume that the sample is taken according to a sampling scheme having inclusion probabilities $\pi_k = Pr(k \in s)$. When π_k is proportional to a positive quantity x_k available over U, and s has a predetermined sample size n, then

$$\pi_k = \frac{n x_k}{\sum_{i=1}^{N} x_i},$$

and the sampling scheme is said to be probability proportional to size (πps).

The Hájek estimator of the population total is defined as

$$\hat{y}_{Hajek} = N \frac{\sum_{k \in s} y_k/\pi_k}{\sum_{k \in s} 1/\pi_k},$$

while the Horvitz-Thompson estimator is

$$\hat{y}_{HT} = \sum_{k \in s} y_k/\pi_k.$$

Särndal, Swenson, and Wretman (1992, p. 182) give several cases for considering the Hájek estimator as 'usually the better estimator' compared to the Horvitz-Thompson estimator when a πps sampling design is used:

a) the $y_k - \bar{y}_U$ tend to be small,

b) the sample size is not fixed,

c) π_k are weakly or negatively correlated with y_k.

Monte Carlo simulation is used here to compare the accuracy of both estimators using a sample size (or the expected value of the sample size) equal to 20. Four cases are considered:

Case 1. y_k is constant for $k = 1, \dots, N$; this case corresponds to the case a) above;

Case 2. Poisson sampling is used to draw a sample s; this case corresponds to the case b) above;

Case 3. y_k are generated using the following model: $x_k = k, \pi_k = nx_k / \sum_{i=1}^{N} x_i, y_k = 1/\pi_k$; this case corresponds to the case c) above;

Case 4. y_k are generated using the following model: $x_k = k, y_k = 5(x_k + \epsilon_k), \epsilon_k \sim N(0, 1/3)$; in this case the Horvitz-Thompson estimator should perform better than the Hájek estimator.

Tillé sampling is used in Cases 1, 3 and 4. Poisson sampling is used in Case 2. The `belgianmunicipalities` dataset is used in Cases 1 and 2 as population, with $x_k = Tot04_k$. In Case 2, the variable of interest is TaxableIncome. The mean square error (MSE) is computed using simulations for each case and estimator. The Hájek estimator should perform better than the Horvitz-Thompson estimator in Cases 1, 2 and 3.

```
> data(belgianmunicipalities)
> attach(belgianmunicipalities)
> # sample size
> n=20
> pik=inclusionprobabilities(Tot04,n)
> N=length(pik)
```

Number of runs (for an accurate result, increase this value to 10000):

```
> sim=10
> ss=ss1=array(0,c(sim,4))
```

Defines the variables of interest:

```
> cat("Case 1\n")
> y1=rep(3,N)
> cat("Case 2\n")
> y2=TaxableIncome
> cat("Case 3\n")
> x=1:N
> pik3=inclusionprobabilities(x,n)
> y3=1/pik3
> cat("Case 4\n")
> epsilon=rnorm(N,0,sqrt(1/3))
> pik4=pik3
> y4=5*(x+epsilon)
```

Monte-Carlo simulation and computation of the Horvitz-Thompson and Hájek estimators:

```
> ht=numeric(4)
> hajek=numeric(4)
> for(i in 1:sim)
+ {
+ cat("Simulation ",i,"\n")
```

```
+ cat("Case 1\n")
+ s=UPtille(pik)
+ ht[1]=HTestimator(y1[s==1],pik[s==1])
+ hajek[1]=Hajekestimator(y1[s==1],pik[s==1],N,type="total")
+ cat("Case 2\n")
+ s1=UPpoisson(pik)
+ ht[2]=HTestimator(y2[s1==1],pik[s1==1])
+ hajek[2]=Hajekestimator(y2[s1==1],pik[s1==1],N,type="total")
+ cat("Case 3\n")
+ ht[3]=HTestimator(y3[s==1],pik3[s==1])
+ hajek[3]=Hajekestimator(y3[s==1],pik3[s==1],N,type="total")
+ cat("Case 4\n")
+ ht[4]=HTestimator(y4[s==1],pik4[s==1])
+ hajek[4]=Hajekestimator(y4[s==1],pik4[s==1],N,type="total")
+ ss[i,]=ht
+ ss1[i,]=hajek
+ }
```

Estimation of the MSE and computation of the ratio MSE_{HT}/MSE_{Hajek} :

```
> #true values
> tv=c(sum(y1),sum(y2),sum(y3),sum(y4))
> for(i in 1:4)
+ {
+ cat("Case ",i,"\n")
+ cat("The mean of the Horvitz-Thompson estimators:",mean(ss[,i])," and the true value:",tv[i],
+ MSE1=var(ss[,i])+(mean(ss[,i])-tv[i])^2
+ cat("MSE Horvitz-Thompson estimator:",MSE1,"\n")
+ cat("The mean of the Hajek estimators:",mean(ss1[,i])," and the true value:",tv[i],"\n")
+ MSE2=var(ss1[,i])+(mean(ss1[,i])-tv[i])^2
+ cat("MSE Hajek estimator:",MSE2,"\n")
+ cat("Ratio of the two MSE:", MSE1/MSE2,"\n")
+ }
```

```
Case  1
The mean of the Horvitz-Thompson estimators: 1575.178  and the true value: 1767
MSE Horvitz-Thompson estimator: 150850.3
The mean of the Hajek estimators: 1767  and the true value: 1767
MSE Hajek estimator: 1.148862e-26
Ratio of the two MSE: 1.313041e+31
Case  2
The mean of the Horvitz-Thompson estimators: 130986110769  and the true value: 121128481686
MSE Horvitz-Thompson estimator: 7.33903e+20
The mean of the Hajek estimators: 121158526642  and the true value: 121128481686
MSE Hajek estimator: 1.816316e+21
Ratio of the two MSE: 0.4040613
Case  3
The mean of the Horvitz-Thompson estimators: 24398670  and the true value: 60436.25
MSE Horvitz-Thompson estimator: 1.24663e+15
```

```
The mean of the Hajek estimators: 1836432  and the true value: 60436.25
MSE Hajek estimator: 4.043144e+12
Ratio of the two MSE: 308.3319
Case  4
The mean of the Horvitz-Thompson estimators: 880514.7  and the true value: 868922.3
MSE Horvitz-Thompson estimator: 242118827
The mean of the Hajek estimators: 111859.5  and the true value: 868922.3
MSE Hajek estimator: 582284075062
Ratio of the two MSE: 0.0004158088

>
```